Marchal’s family

Inclined co-orbitals in the three-body problem

Alexandre Prieur

With Philippe Robutel & Jacques Fejoz

22/01/2026

Introduction

The Lagrange relative equilibrium

Lagrange relative equilibrium of the 3BP

The Lagrange relative equilibrium

Lagrange relative equilibrium of the 3BP

Elliptic if

$$\frac{m_0 * m_1 + m_0 * m_2 + m_1 * m_2}{\left(m_0 + m_1 + m_2\right)^2} > \frac{1}{27}$$

The Lagrange relative equilibrium

Lagrange relative equilibrium of the 3BP

Elliptic if

$$\frac{m_0 * m_1 + m_0 * m_2 + m_1 * m_2}{\left(m_0 + m_1 + m_2\right)^2} > \frac{1}{27}$$

$$\begin{cases} (1 - \varepsilon, \varepsilon, 0) & \rightarrow \varepsilon_G \approx 0.0385 \\ (1 - 2\varepsilon, \varepsilon, \varepsilon) & \rightarrow \varepsilon_G \approx 0.0191 \end{cases}$$

The Lagrange relative equilibrium – Lyapunov families

Lyapunov families emerging from the elliptic directions of Lagrange

Homographic family

The Lagrange relative equilibrium – Lyapunov families

Lyapunov families emerging from the elliptic directions of Lagrange

Homographic family

Vertical family

The Lagrange relative equilibrium – Lyapunov families

Lyapunov families emerging from the elliptic directions of Lagrange

Homographic family

Vertical family

Circular librations

The Lagrange relative equilibrium – Lyapunov families

Lyapunov families emerging from the elliptic directions of Lagrange

Homographic family	Vertical family
Circular librations	Anti-Lagrange

The Lagrange relative equilibrium – Lyapunov families

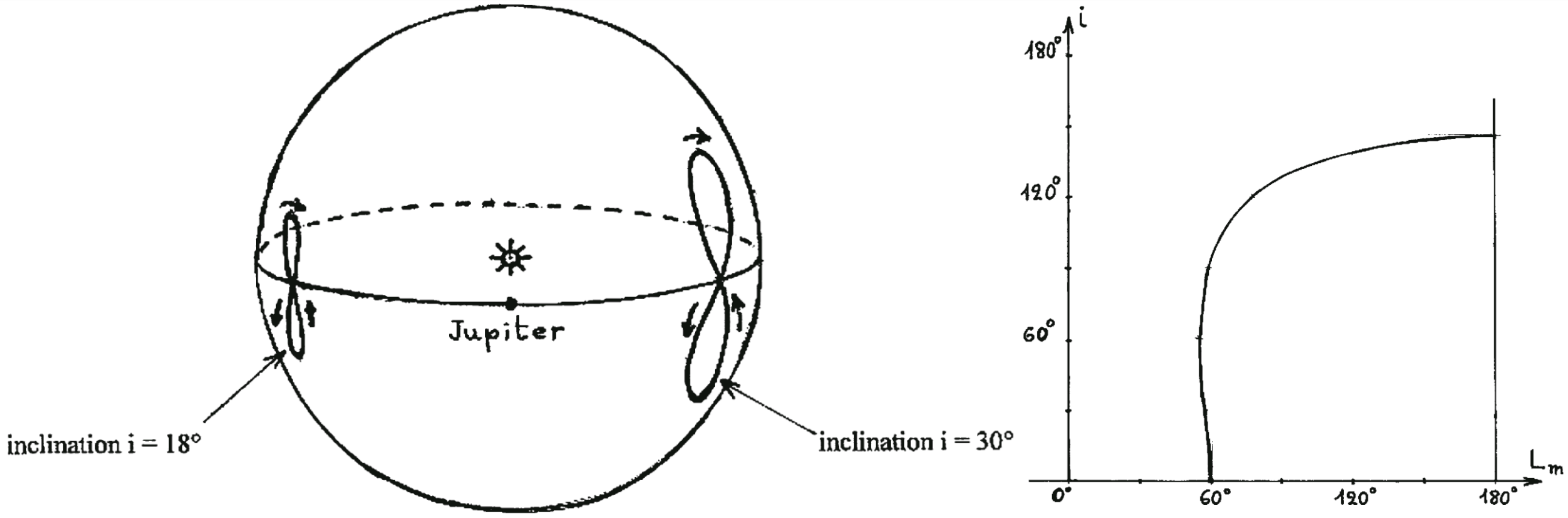
Lyapunov families emerging from the elliptic directions of Lagrange

Homographic family	Vertical family
Circular librations	Anti-Lagrange

The vertical family – equal masses: P_{12}

- (Marchal, 1990) – P_{12} near Lagrange
- (Chenciner and Montgomery, 2000) – Eight and P_{12}
- (Chenciner and Fejoz, 2008) – uniqueness of P_{12} near Lagrange
- (Calleja et al., 2024) – global uniqueness of P_{12}

The vertical family – small masses



- (Marchal, 2009) – $\mathcal{VF}_L$ in the average restricted problem
  - Already a conjecture – part of a three-parameter family, including P_{12}!
- (Leleu, 2016) – numerical hints of a stable region in the full planetary problem

Questions

Q1: Does the family persist in the non-restricted/non-average problem?
Q2: What is the stability of the orbits along this family?
Q3: Does it have an impact on the global stability of the co-orbital region?
Q4: How is it linked to P_{12}?

Questions

Q1: Does the family persist in the non-restricted/non-average problem?

Q2: What is the stability of the orbits along this family?

Q3: Does it have an impact on the global stability of the co-orbital region?

Q4: How is it linked to P_{12}?

$\mathcal{VF}_L$ in the average circular planetary problem

Frame of reference

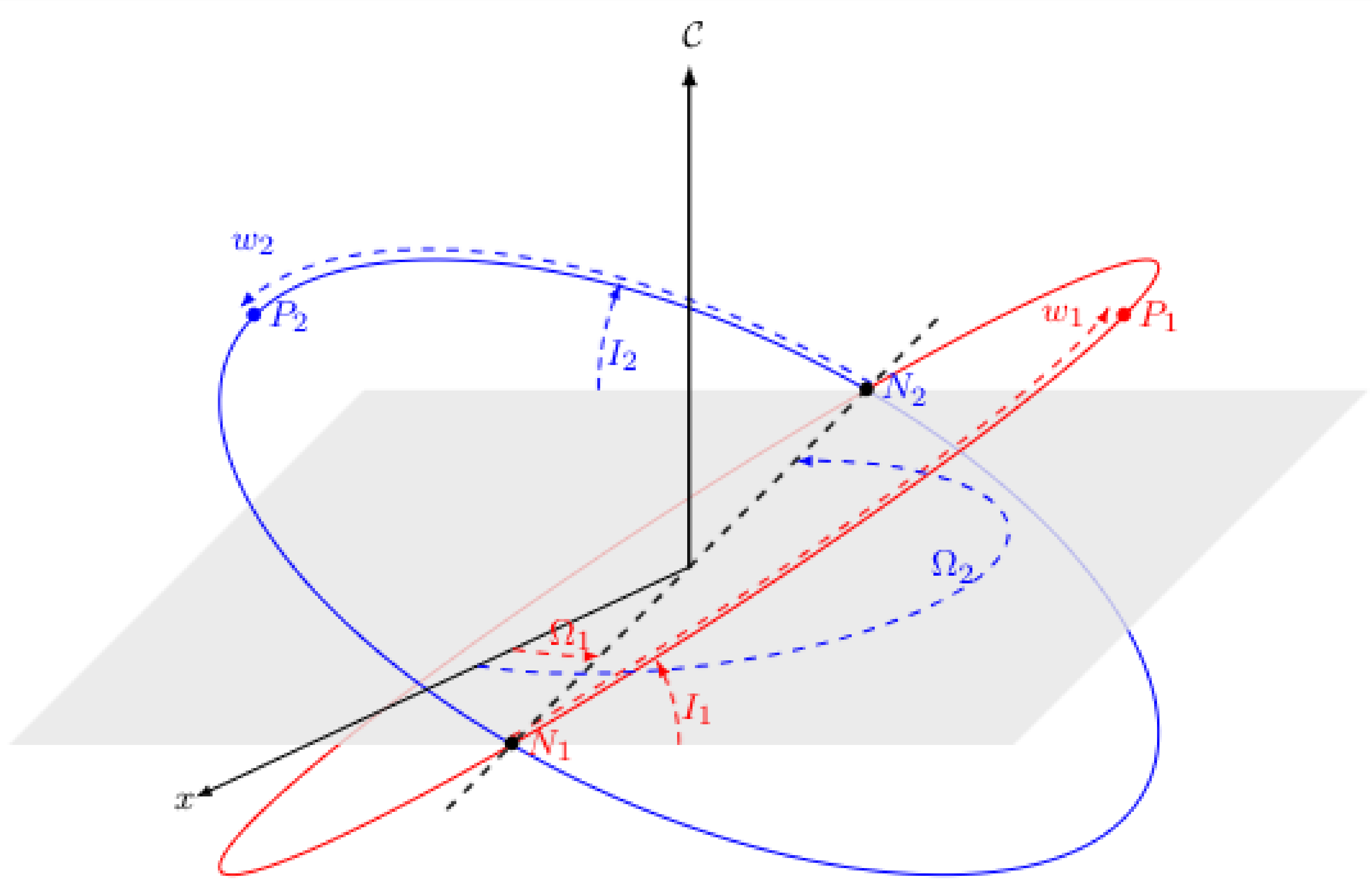
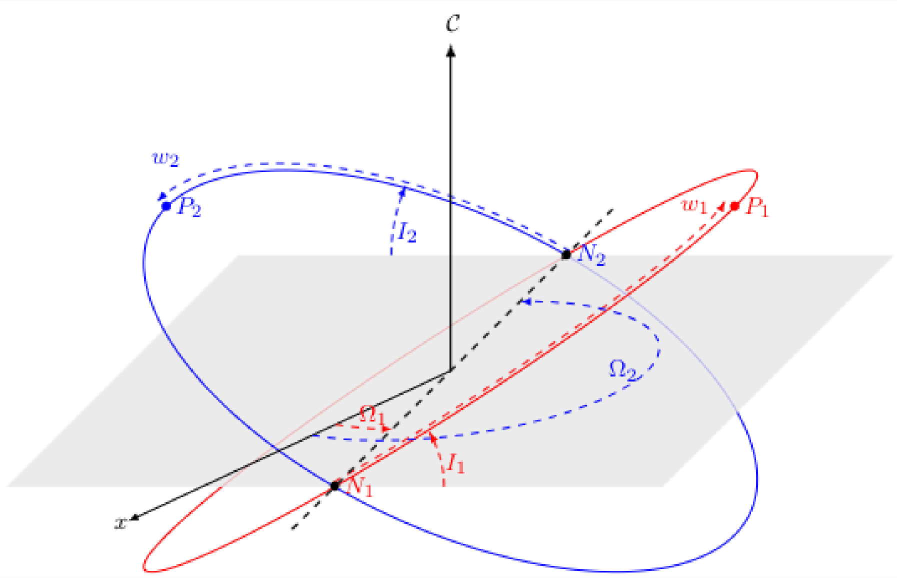


Figure 2: Canonical heliocentric : heliocentric positions, barycentric momenta; $(Oxy) \perp \boldsymbol{C}$

Complex Poincaré coordinates



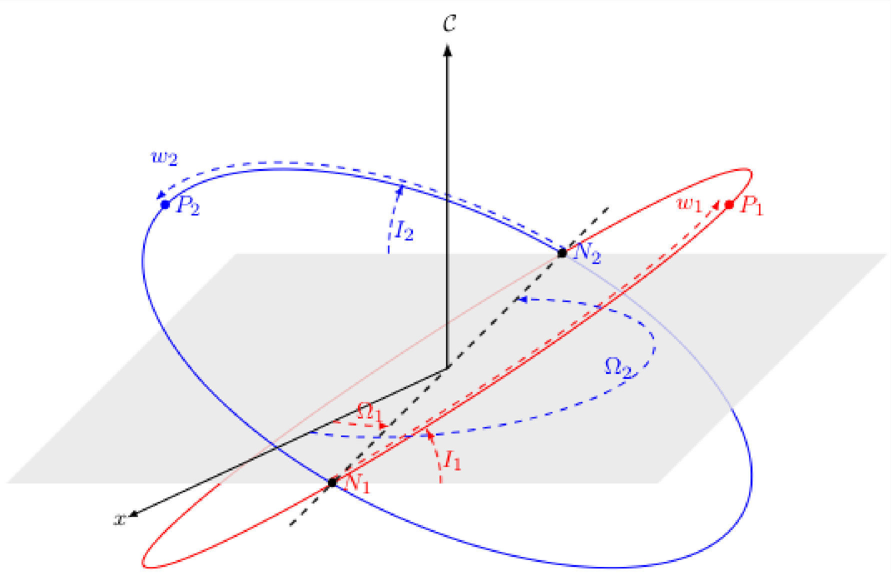
$$\beta_j = \frac{m_0 m_j}{m_0 + m_j}, \mu_j = \mathcal{G}\left(m_0 + m_j\right)$$

$$\Lambda_j = \beta_j \sqrt{\mu_j a_j} \qquad \tilde{\lambda}_j = M_j + \omega_j$$

$$x_j = \underbrace{\sqrt{\Lambda_j - G_j}}_{\sim e_j} \exp\left(i\omega_j\right) \qquad \tilde{x}_j = -i\bar{x}_j$$

$$\Phi_j = \mathbf{G_1}_z \pm \mathbf{G_2}_z \qquad \varphi_j = \frac{\Omega_1 \pm \Omega_2}{2}$$

Jacobi reduction



$$\Omega_2 - \Omega_1 = \pi = 2\varphi_2$$

$\Rightarrow$ φ_1 and Φ_1 ignorable

$$\Phi_1 = C_z = C = \text{constant}$$

$\Rightarrow$ φ_2 ignorable

Jacobi reduction – consequences

We are left with 4 degrees of freedom and one parameter C

Jacobi reduction – consequences

We are left with 4 degrees of freedom and one parameter C

As we do not track the nodes, they are at a fixed position in this frame: draconic frame

Jacobi reduction – consequences

We are left with 4 degrees of freedom and one parameter C

As we do not track the nodes, they are at a fixed position in this frame: draconic frame

→ We haven't lost the nodes or the inclinations!

$$\begin{cases} J := I_1 + I_2, \quad \cos(J) = \frac{C^2 - G_1^2 - G_2^2}{2G_1G_2}, \quad C\cos\left(I_j\right) = G_j + G_{1-j}\cos(J) \\ \dot{\Omega}_j = \frac{\partial H}{\partial C} \end{cases}$$

Co-orbital motion in the planetary case

- Planetary problem

$$\left(1 - \varepsilon\left(m_1 + m_2\right), \varepsilon m_1, \varepsilon m_2\right),\ \varepsilon \ll 1$$

- ε-neighborhood of co-orbital motions

$$a_1, a_2 = a^\star + \mathcal{O}(\varepsilon)$$

Co-orbital motion in the planetary case

- Planetary problem

$$\left(1 - \varepsilon\left(m_1 + m_2\right), \varepsilon m_1, \varepsilon m_2\right), \ \varepsilon \ll 1$$

- ε-neighborhood of co-orbital motions

$$a_1, a_2 = a^\star + \mathcal{O}(\varepsilon) = \frac{{\Lambda_j^\star}^2}{\beta_j^2 \mu_j}$$

Co-orbital motion in the planetary case

- Planetary problem

$$\big(1 - \varepsilon(m_1 + m_2), \varepsilon m_1, \varepsilon m_2\big), \ \varepsilon \ll 1$$

- ε-neighborhood of co-orbital motions

$$a_1, a_2 = a^\star + \mathcal{O}(\varepsilon) = \frac{{\Lambda_j^\star}^2}{\beta_j^2 \mu_j}$$

We set

$$\begin{cases} Z_1 = \Lambda_1 - \Lambda_1^\star & \zeta_1 = \Delta\tilde{\lambda}_j + \pi \\ Z_2 = \sum \Lambda_j - \sum \Lambda_j^\star & \zeta_2 = \tilde{\lambda}_2 \end{cases}$$

Truncation in Z_j

We stay in an $\sqrt{\varepsilon}$-neighbourhood of $\Lambda_j^\star$: $Z_j = \mathcal{O}\left(\varepsilon^{\frac{1}{2}}\right)$

Truncation in Z_j

We stay in an $\sqrt{\varepsilon}$-neighbourhood of $\Lambda_j^\star$: $Z_j = \mathcal{O}\left(\varepsilon^{\frac{1}{2}}\right)$

$$H(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{x},\tilde{\boldsymbol{x}},C) = \underbrace{H_K(\boldsymbol{Z})}_{\text{Keplerian part}} + \underbrace{\varepsilon H_P(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{x},\tilde{\boldsymbol{x}},C)}_{\text{Perturbation}}$$

Truncation in Z_j

We stay in an $\sqrt{\varepsilon}$-neighbourhood of $\Lambda_j^\star$: $Z_j = \mathcal{O}\left(\varepsilon^{\frac{1}{2}}\right)$

$$H(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{x},\tilde{\boldsymbol{x}},C) = \underbrace{H_K(\boldsymbol{Z})}_{\text{Keplerian part}} + \underbrace{\varepsilon H_P(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{x},\tilde{\boldsymbol{x}},C)}_{\text{Perturbation}}$$

Keplerian part

- Order 1 and ≥ 3 of order $\varepsilon^{\frac{3}{2}}$
- At $\mathcal{O}(\varepsilon)$: orders 0 and 2 remain

Truncation in Z_j

We stay in an $\sqrt{\varepsilon}$-neighbourhood of $\Lambda_j^\star$: $Z_j = \mathcal{O}\left(\varepsilon^{\frac{1}{2}}\right)$

$$H(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{x},\tilde{\boldsymbol{x}},C) = \underbrace{H_K(\boldsymbol{Z})}_{\text{Keplerian part}} + \underbrace{\varepsilon H_P(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{x},\tilde{\boldsymbol{x}},C)}_{\text{Perturbation}}$$

Keplerian part

- Order 1 and ≥ 3 of order $\varepsilon^{\frac{3}{2}}$
- At $\mathcal{O}(\varepsilon)$: orders 0 and 2 remain

Perturbation

- Perturbation is already of order ε
- Orders ≥ 1 of order > ε
- At $\mathcal{O}(\varepsilon)$: order 0

Truncation in Z_j

We stay in an $\sqrt{\varepsilon}$-neighbourhood of $\Lambda_j^\star$: $Z_j = \mathcal{O}\left(\varepsilon^{\frac{1}{2}}\right)$

$$H(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{x},\tilde{\boldsymbol{x}},C) = \underbrace{H_K(\boldsymbol{Z})}_{\text{Keplerian part}} + \underbrace{\varepsilon H_P(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{x},\tilde{\boldsymbol{x}},C)}_{\text{Perturbation}}$$

Keplerian part

- Order 1 and ≥ 3 of order $\varepsilon^{\frac{3}{2}}$
- At $\mathcal{O}(\varepsilon)$: orders 0 and 2 remain

Perturbation

- Perturbation is already of order ε
- Orders ≥ 1 of order > ε
- At $\mathcal{O}(\varepsilon)$: order 0

$$H(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{x},\tilde{\boldsymbol{x}},C) = H_K^{(2)}(\boldsymbol{Z}) + \varepsilon H_P(\boldsymbol{0},\boldsymbol{\zeta},\boldsymbol{x},\tilde{\boldsymbol{x}},C) + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$$

Averaging

$$H(\boldsymbol{Z}, \boldsymbol{\zeta}, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C) = H_K^{(2)}(\boldsymbol{Z}) + \varepsilon H_P(\boldsymbol{0}, \boldsymbol{\zeta}, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C) + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$$

Averaging

$$H(\boldsymbol{Z}, \boldsymbol{\zeta}, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C) = H_K^{(2)}(\boldsymbol{Z}) + \varepsilon H_P(\boldsymbol{0}, \boldsymbol{\zeta}, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C) + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$$

$$\downarrow \text{(averaging)}$$

$$F\left(\boldsymbol{Z}, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) = H_K^{(2)}(\boldsymbol{Z}) + \varepsilon F_P\left(\boldsymbol{0}, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$$

Averaging

$$H(\boldsymbol{Z}, \boldsymbol{\zeta}, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C) = H_K^{(2)}(\boldsymbol{Z}) + \varepsilon H_P(\boldsymbol{0}, \boldsymbol{\zeta}, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C) + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$$

$$\downarrow \text{(averaging)}$$

$$F\left(\boldsymbol{Z}, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) = H_K^{(2)}(\boldsymbol{Z}) + \varepsilon F_P\left(\boldsymbol{0}, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$$

$$\downarrow (\mathrm{Z}_2 \text{ constant})$$

$$F\left(\mathrm{Z}_1, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) = \alpha \mathrm{Z}_1^2 + \varepsilon F_P\left(\boldsymbol{0}, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$$

Averaging

$$H(\boldsymbol{Z}, \boldsymbol{\zeta}, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C) = H_K^{(2)}(\boldsymbol{Z}) + \varepsilon H_P(\boldsymbol{0}, \boldsymbol{\zeta}, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C) + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$$

$$\downarrow \text{(averaging)}$$

$$F\left(\boldsymbol{Z}, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) = H_K^{(2)}(\boldsymbol{Z}) + \varepsilon F_P\left(\boldsymbol{0}, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$$

$$\downarrow (\mathrm{Z}_2 \text{ constant})$$

$$F\left(\mathrm{Z}_1, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) = \alpha \mathrm{Z}_1^2 + \varepsilon F_P\left(\boldsymbol{0}, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$$

$$\downarrow$$

$$\overline{F}\left(\mathrm{Z}_1, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right) = \alpha \mathrm{Z}_1^2 + \varepsilon F_P\left(\boldsymbol{0}, \zeta_1, \boldsymbol{x}, \tilde{\boldsymbol{x}}, C\right)$$

Circular case

The expansion of $\bar{F}$ in powers of $(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ is even!

The manifold $\boldsymbol{x} = \tilde{\boldsymbol{x}} = \mathbf{0}$ is invariant under the flow.

Circular case

The expansion of $\bar{F}$ in powers of $(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ is even!

The manifold $\boldsymbol{x} = \tilde{\boldsymbol{x}} = \boldsymbol{0}$ is invariant under the flow.

$$\bar{F}\big(\boldsymbol{Z}, \zeta_1, \boldsymbol{0}, \boldsymbol{0}, C\big) = \alpha Z_1^2 + \varepsilon F_P\big(\boldsymbol{0}, \zeta_1, \boldsymbol{0}, \boldsymbol{0}, C\big)$$

Circular case

The expansion of $\bar{F}$ in powers of $(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ is even!

The manifold $\boldsymbol{x} = \tilde{\boldsymbol{x}} = \mathbf{0}$ is invariant under the flow.

$$\bar{F}\big(\boldsymbol{Z}, \zeta_1, \mathbf{0}, \mathbf{0}, C\big) = \alpha Z_1^2 + \varepsilon F_P\big(\mathbf{0}, \zeta_1, \mathbf{0}, \mathbf{0}, C\big)$$

$$\bar{F}\big(Z_1, \zeta_1, C\big) = \alpha Z_1^2 + \varepsilon \beta F_1\big(\zeta_1, C\big)$$

β depending on m_1, m_2; F_1 independant of masses

A one-parameter family: $\mathcal{VF}_L$

$$\overline{F}\left(Z_1, \zeta_1, C\right) = \alpha Z_1^2 + \varepsilon\beta F_1\left(\zeta_1, C\right)$$

A one-parameter family: $\mathcal{V}\mathcal{F}_L$

$$\bar{F}(Z_1, \zeta_1, C) = \alpha Z_1^2 + \varepsilon\beta F_1(\zeta_1, C)$$

For a fixed C_0, one-parameter family satisfying

$$Z_1 = 0, \quad \zeta_1 = \zeta(C_0), \quad \frac{\partial F_1}{\partial \zeta_1}(\zeta(C_0), C_0) = 0$$

A one-parameter family: $\mathcal{VF}_L$

$$\bar{F}(Z_1, \zeta_1, C) = \alpha Z_1^2 + \varepsilon\beta F_1(\zeta_1, C)$$

For a fixed C_0, one-parameter family satisfying

$$Z_1 = 0, \quad \zeta_1 = \zeta(C_0), \quad \frac{\partial F_1}{\partial \zeta_1}(\zeta(C_0), C_0) = 0$$

F_1 independant of masses! $\Rightarrow$ Study valid for all small masses

A one-parameter family: $\mathcal{VF}_L$

$$\bar{F}(Z_1, \zeta_1, C) = \alpha Z_1^2 + \varepsilon\beta F_1(\zeta_1, C)$$

For a fixed C_0, one-parameter family satisfying

$$Z_1 = 0, \quad \zeta_1 = \zeta(C_0), \quad \frac{\partial F_1}{\partial \zeta_1}(\zeta(C_0), C_0) = 0$$

F_1 independant of masses! ⇒ Study valid for all small masses

- Analytical approximation
- Or numerical search

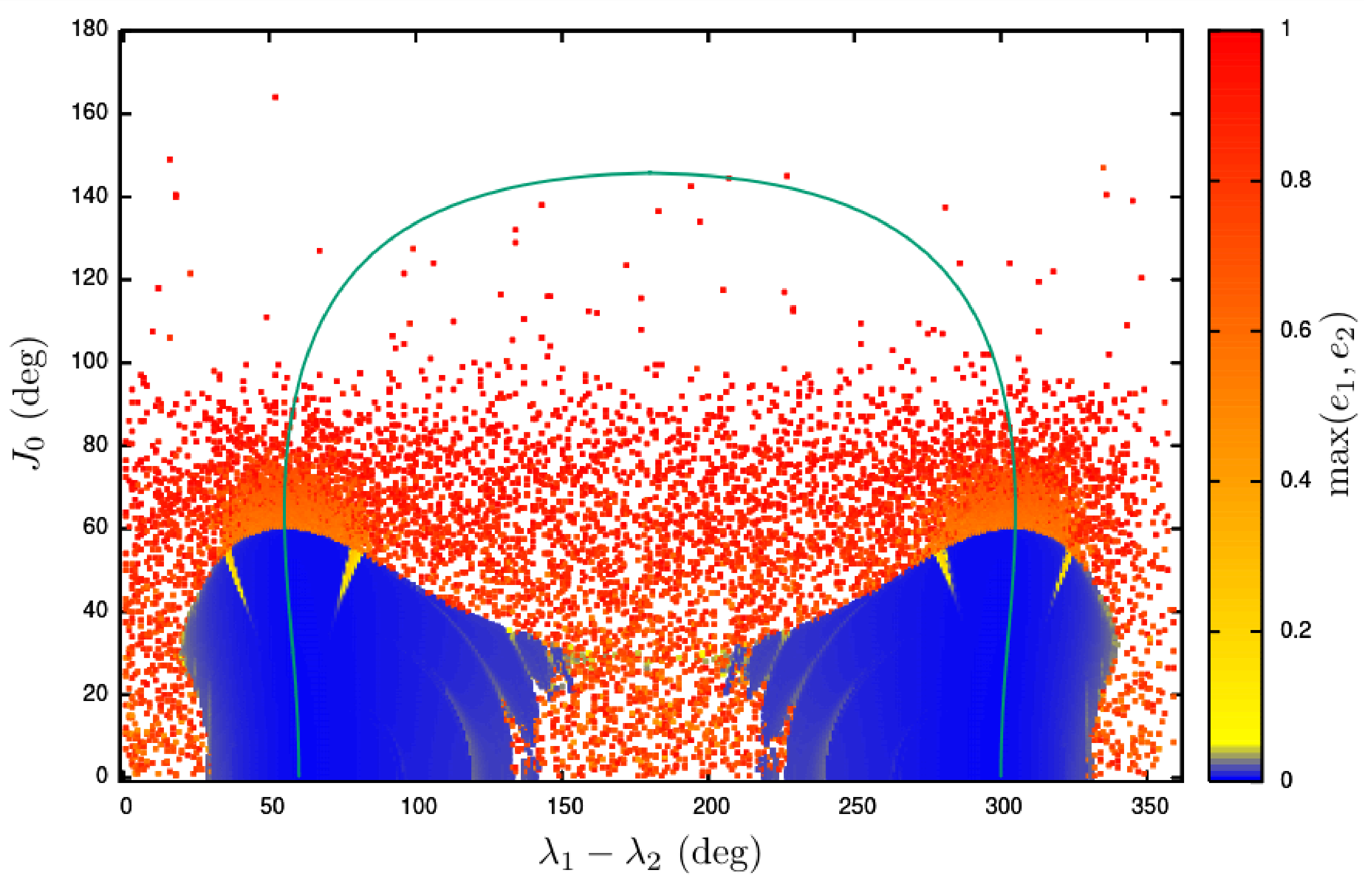
What can we say?

Analytical: 20^{th} degree expansion in $\sin\left(\frac{J_{\max}}{2}\right)$

What can we say?

Analytical: 20^{th} degree expansion in $\sin\left(\frac{J_{\max}}{2}\right)$

Stability: $\mathcal{VF}_L$ is stable in the average **circular** problem



Numerical search in the full problem

A new set of coordinates

- Numerical integration: need coordinates with explicit expression of the Hamiltonian/vector field
- The Jacobi reduction should be explicit in these coordinates

A new set of coordinates

- Numerical integration: need coordinates with explicit expression of the Hamiltonian/vector field
- The Jacobi reduction should be explicit in these coordinates

Hill coordinates!

$$\begin{cases} R_j & r_j \\ G_j & w_j \\ G_j \cos(I_j) & \Omega_j \end{cases}$$

A new set of coordinates

- Numerical integration: need coordinates with explicit expression of the Hamiltonian/vector field
- The Jacobi reduction should be explicit in these coordinates

Hill coordinates!

$$\begin{cases} R_j & r_j \\ G_j & w_j \\ \cancel{G_j \cos(I_j)} & \cancel{\Omega_j} \end{cases}$$

Lagrange in Hill coordinates

Lagrange in Hill coordinates:

$$r_j = a^\star \quad R_j = \frac{\sqrt{3}\omega^\star a^\star m_1 m_2}{2M}$$

$$w_1 - w_2 = \frac{5\pi}{3} \quad G_j = \frac{\omega^\star {a^\star}^2 m_j \left(2m_0 + m_{1-j}\right)}{2M}$$

Lagrange in Hill coordinates

Lagrange in Hill coordinates:

$$r_j = a^\star \quad R_j = \frac{\sqrt{3}\omega^\star a^\star m_1 m_2}{2M}$$

$$w_1 - w_2 = \frac{5\pi}{3} \quad G_j = \frac{\omega^\star {a^\star}^2 m_j \left(2m_0 + m_{1-j}\right)}{2M}$$

These are not action-angle variables: Lagrange isn't a fixed point. But this allows for an explicit hamiltonian

How to search for periodic orbits?

Looking for zeros of

$$f : \begin{cases} \mathbb{R}^n & \rightarrow \mathbb{R}^{\mathrm{n}} \\ \\ (\boldsymbol{x}; C) & \mapsto \Phi_T\,(\boldsymbol{x}; C) - \mathrm{x} \end{cases}$$

How to search for periodic orbits?

Looking for zeros of

$$f : \begin{cases} \mathbb{R}^n & \to \mathbb{R}^{\mathrm{n}} \\ \\ (\boldsymbol{x}; C) & \mapsto \Phi_T\,(\boldsymbol{x}; C) - \mathrm{x} \end{cases}$$

But the zeros aren't locally unique!

How to search for periodic orbits?

Looking for zeros of

$$f : \begin{cases} \mathbb{R}^n & \to \mathbb{R}^{\mathrm{n}+\mathrm{m}} \\ (\boldsymbol{x}; C) & \mapsto \begin{pmatrix} \Phi_T(\boldsymbol{x};C)-\boldsymbol{x} \\ \sigma(\boldsymbol{x};C) \end{pmatrix} \end{cases}$$

But the zeros aren't locally unique!

Solution: add sections

How to search for periodic orbits?

In our case:

$$\begin{cases} \sigma_1(\boldsymbol{x}; C) = w_1 + w_2 \\ \sigma_2(\boldsymbol{x}; C) = \mathrm{J} - J_0 \end{cases}$$

Selecting a point on the orbit

How to search for periodic orbits?

In our case:

$$\begin{cases} \sigma_1(\boldsymbol{x}; C) = w_1 + w_2 & \text{Selecting a point on the orbit} \\ \sigma_2(\boldsymbol{x}; C) = \mathrm{J} - J_0 & \text{Selecting an orbit in the family} \end{cases}$$

How to search for periodic orbits?

In our case:

$$\begin{cases} \sigma_1(\boldsymbol{x}; C) = w_1 + w_2 & \text{Selecting a point on the orbit} \\ \sigma_2(\boldsymbol{x}; C) = \mathrm{J} - J_0 & \text{Selecting an orbit in the family} \end{cases}$$

Focusing on the case of two equal masses: $(1 - 2\varepsilon, \varepsilon, \varepsilon)$

How to search for periodic orbits?

In our case:

$$\begin{cases} \sigma_1(\boldsymbol{x}; C) = w_1 + w_2 & \text{Selecting a point on the orbit} \\ \sigma_2(\boldsymbol{x}; C) = \mathrm{J} - J_0 & \text{Selecting an orbit in the family} \end{cases}$$

Focusing on the case of two equal masses: $(1 - 2\varepsilon, \varepsilon, \varepsilon)$

Rootfinding algorithm: multi-dimensional Newton-Raphson method

How to search for periodic orbits?

In our case:

$$\begin{cases} \sigma_1(\boldsymbol{x}; C) = w_1 + w_2 & \text{Selecting a point on the orbit} \\ \sigma_2(\boldsymbol{x}; C) = \mathrm{J} - J_0 & \text{Selecting an orbit in the family} \end{cases}$$

Focusing on the case of two equal masses: $(1 - 2\varepsilon, \varepsilon, \varepsilon)$

Rootfinding algorithm: multi-dimensional Newton-Raphson method

Numerical integration with `taylor` software (Jorba and Zou, 2005) via `TaylorIntegration.jl` (in julia (Bezanson et al., 2017))

Small masses: $\varepsilon = 10^{-3}$

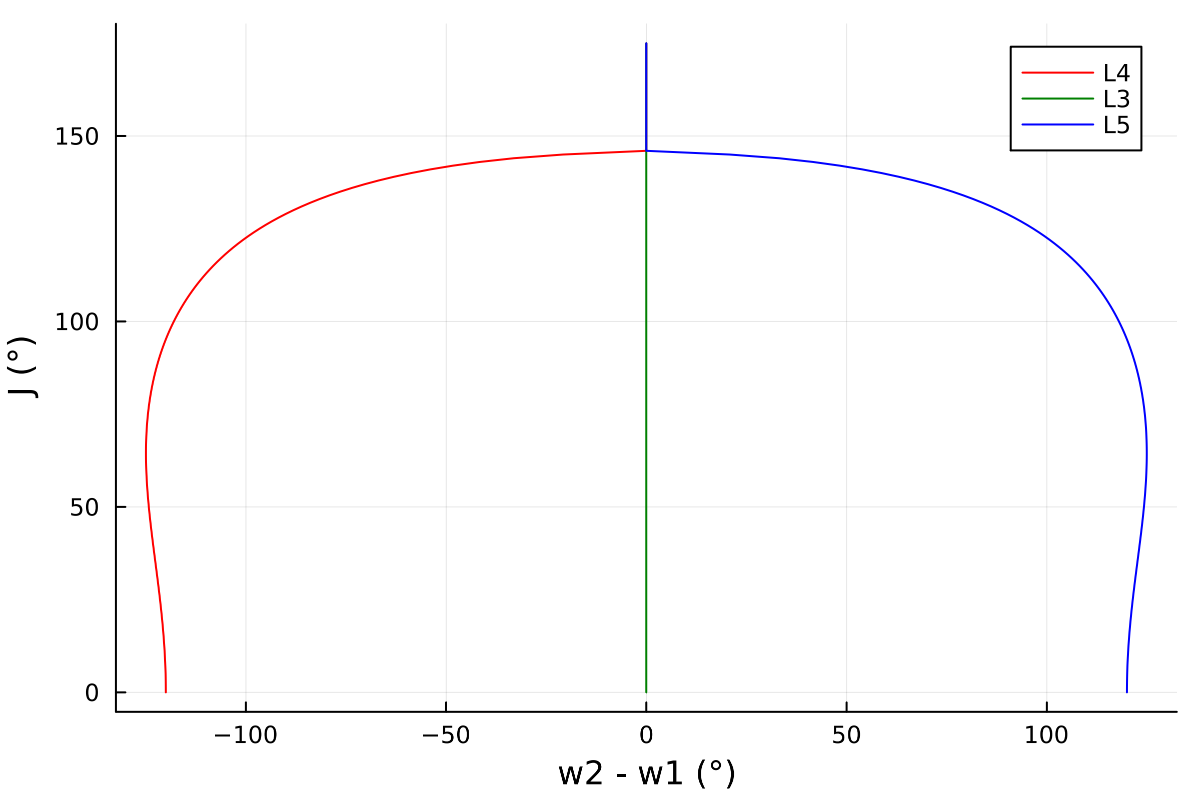
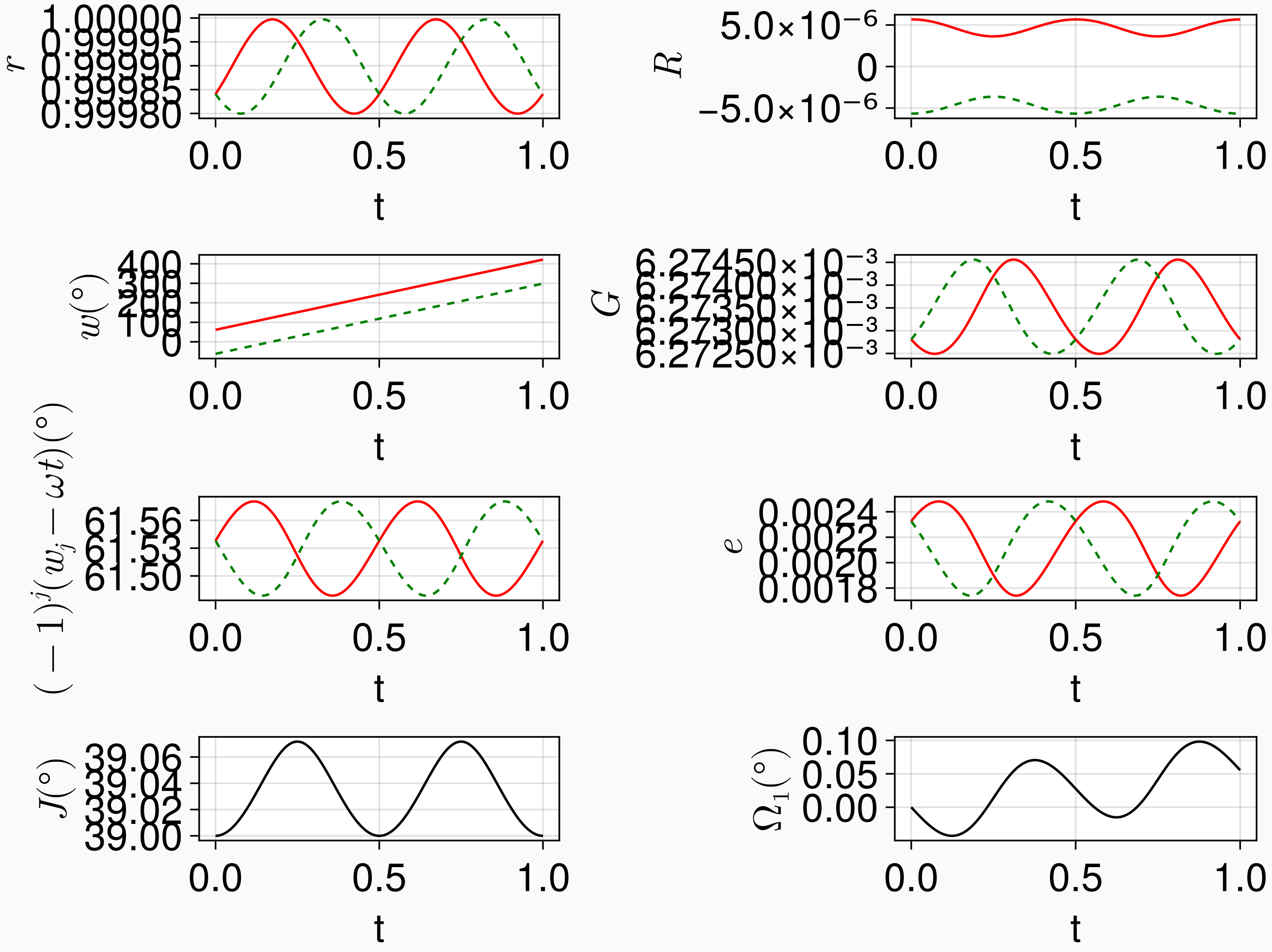


Figure 7: $\mathcal{VF}_L$ from L_4 and L_5, $\mathcal{VF}_E$ from L_3

★

Coordinates along trajectory at $\varepsilon = 10^{-3}$

Coordinates at $J = 40°$



Symetries

Half-time symetry

$$\forall x \in \mathcal{VF}_L, \Phi_{\frac{T}{2}}(\boldsymbol{x}; C) = -\boldsymbol{x}$$

$$\Rightarrow f : (\boldsymbol{x}; C) \mapsto \begin{pmatrix} \Phi_{\frac{T}{2}}(\boldsymbol{x}; C) + \boldsymbol{x} \\ \sigma_1(\boldsymbol{x}; C) \\ \sigma_2(\boldsymbol{x}; C) \end{pmatrix}$$

Cuts integration time in half

Time-reversal symetry

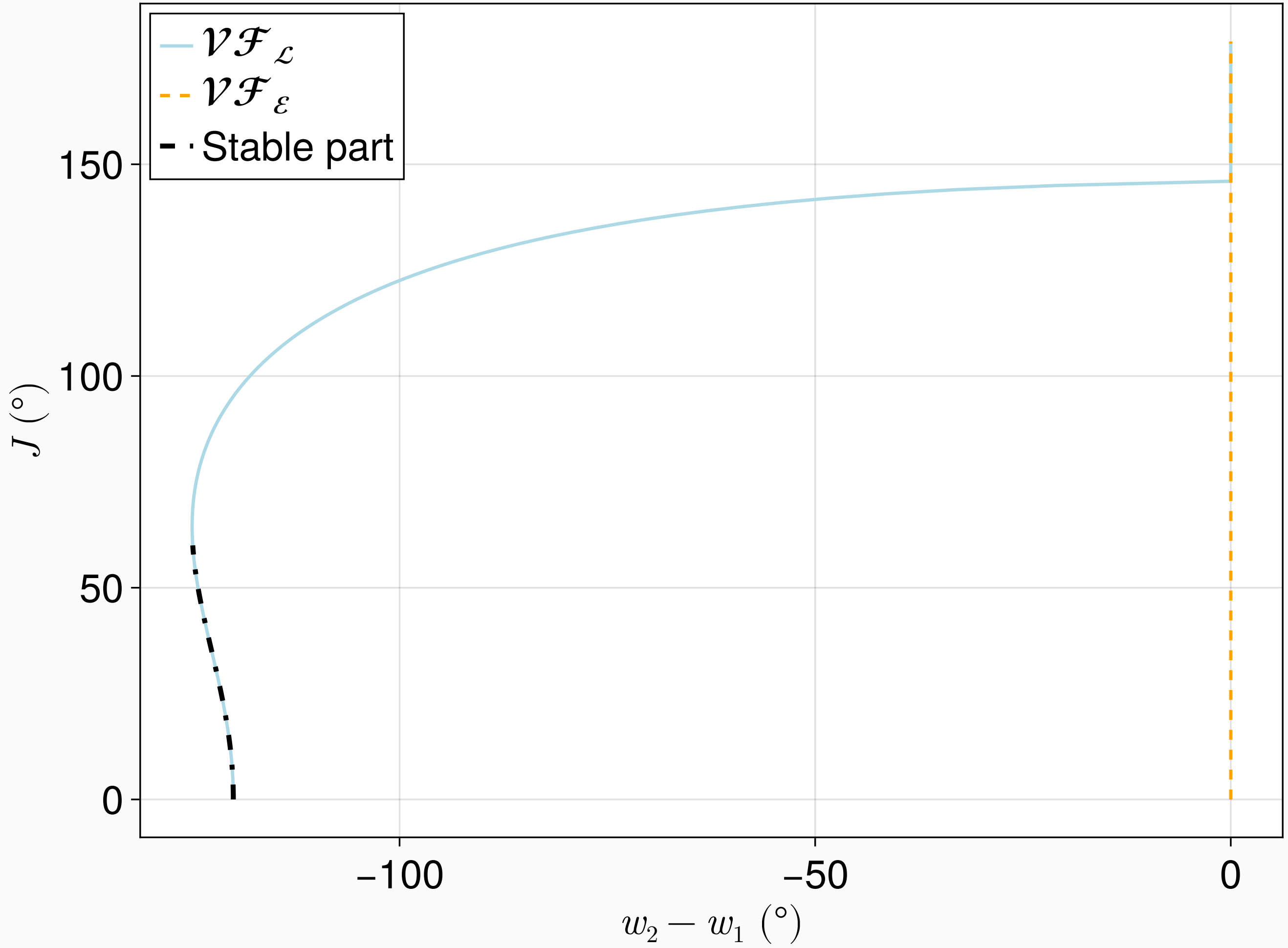
$$\forall x \in \mathcal{VF}_L, t \in \mathbb{R},$$

$$\Phi_{-t}(\boldsymbol{x}; C) = R_y(\pi) p_{1 \leftrightarrow 2} \Phi_t(\boldsymbol{x}; C)$$

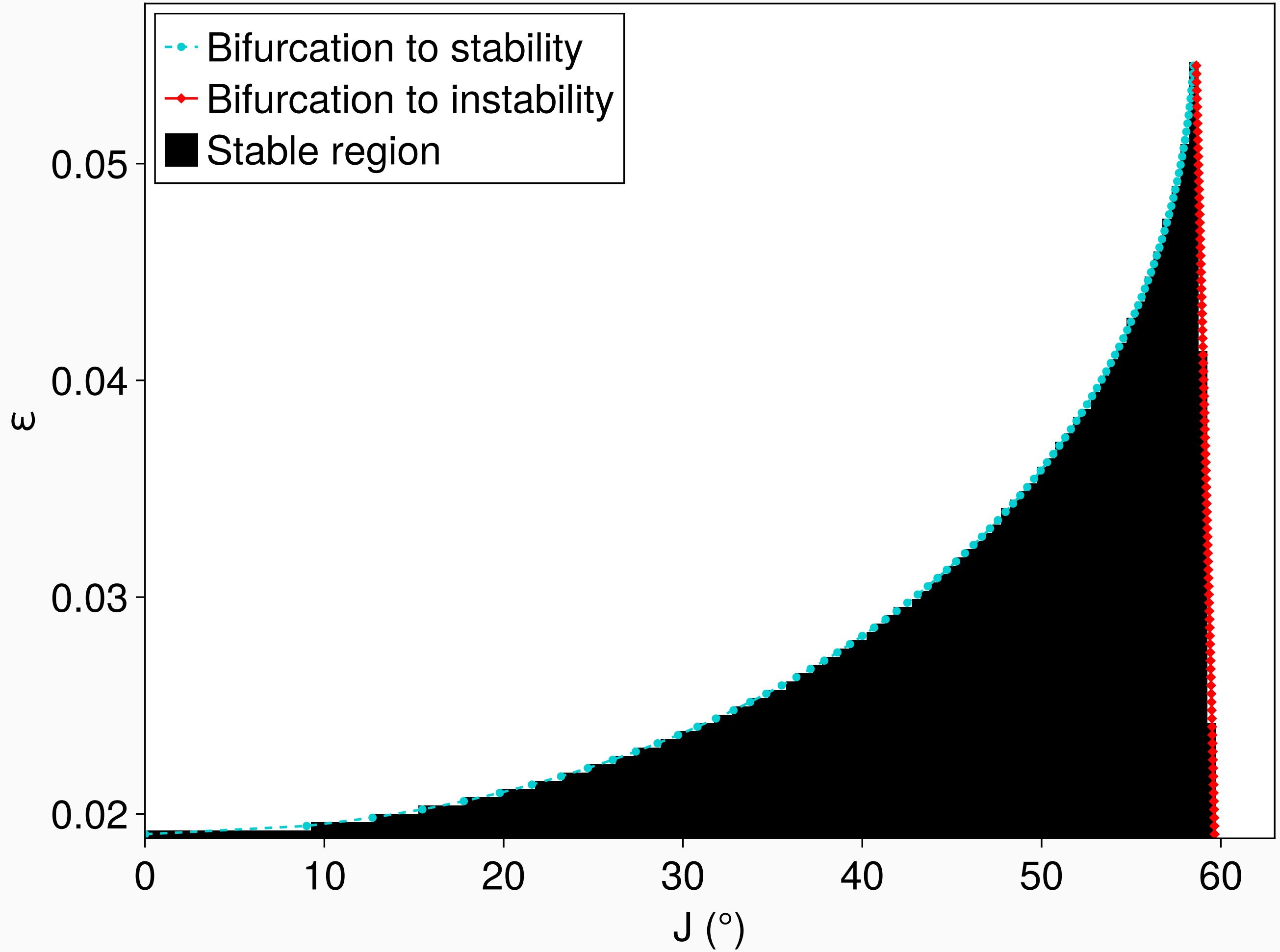
Allows to add sections

$$\sigma_3(\boldsymbol{x}; C) = \begin{pmatrix} r_1 - r_2 \\ R_1 + R_2 \\ G_1 - G_2 \end{pmatrix}$$

Stability at $\varepsilon = 10^{-3}$

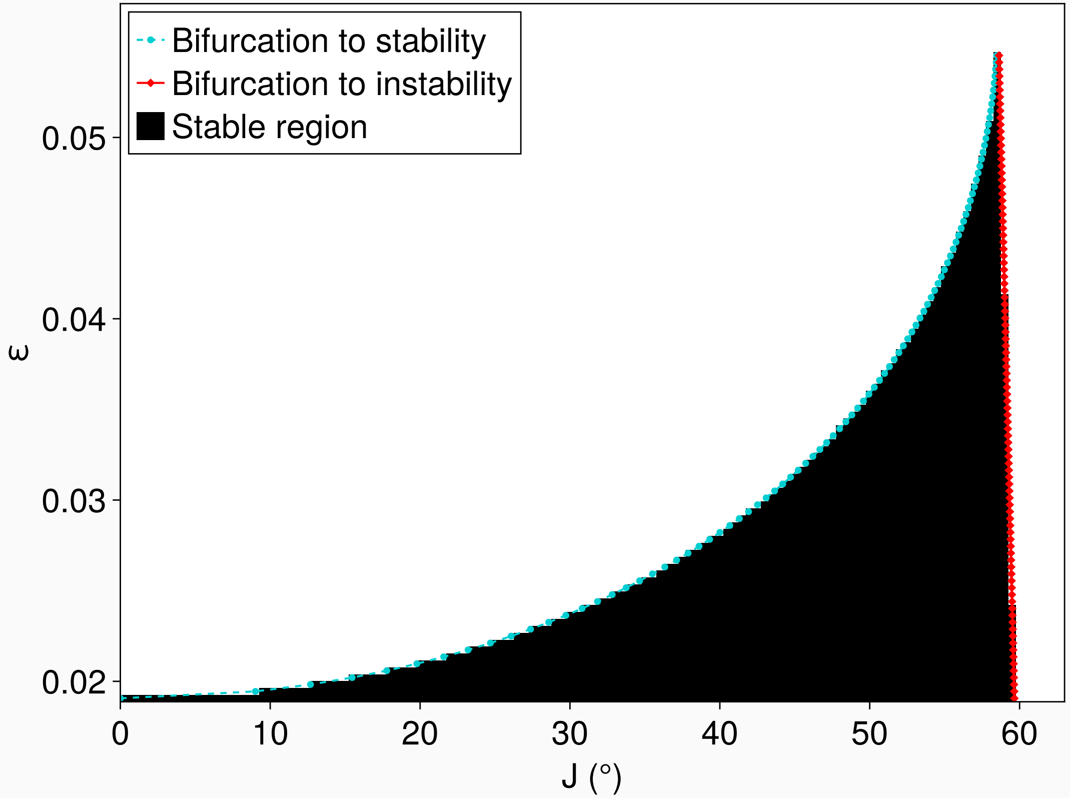


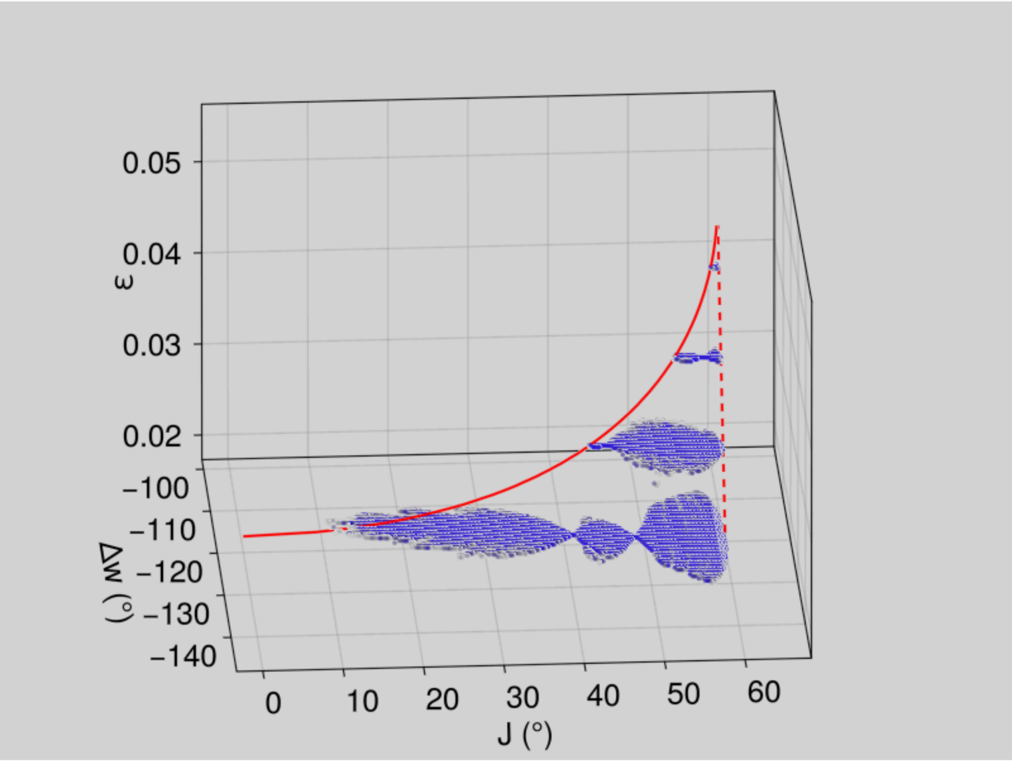
Increasing masses: above Gascheau's value



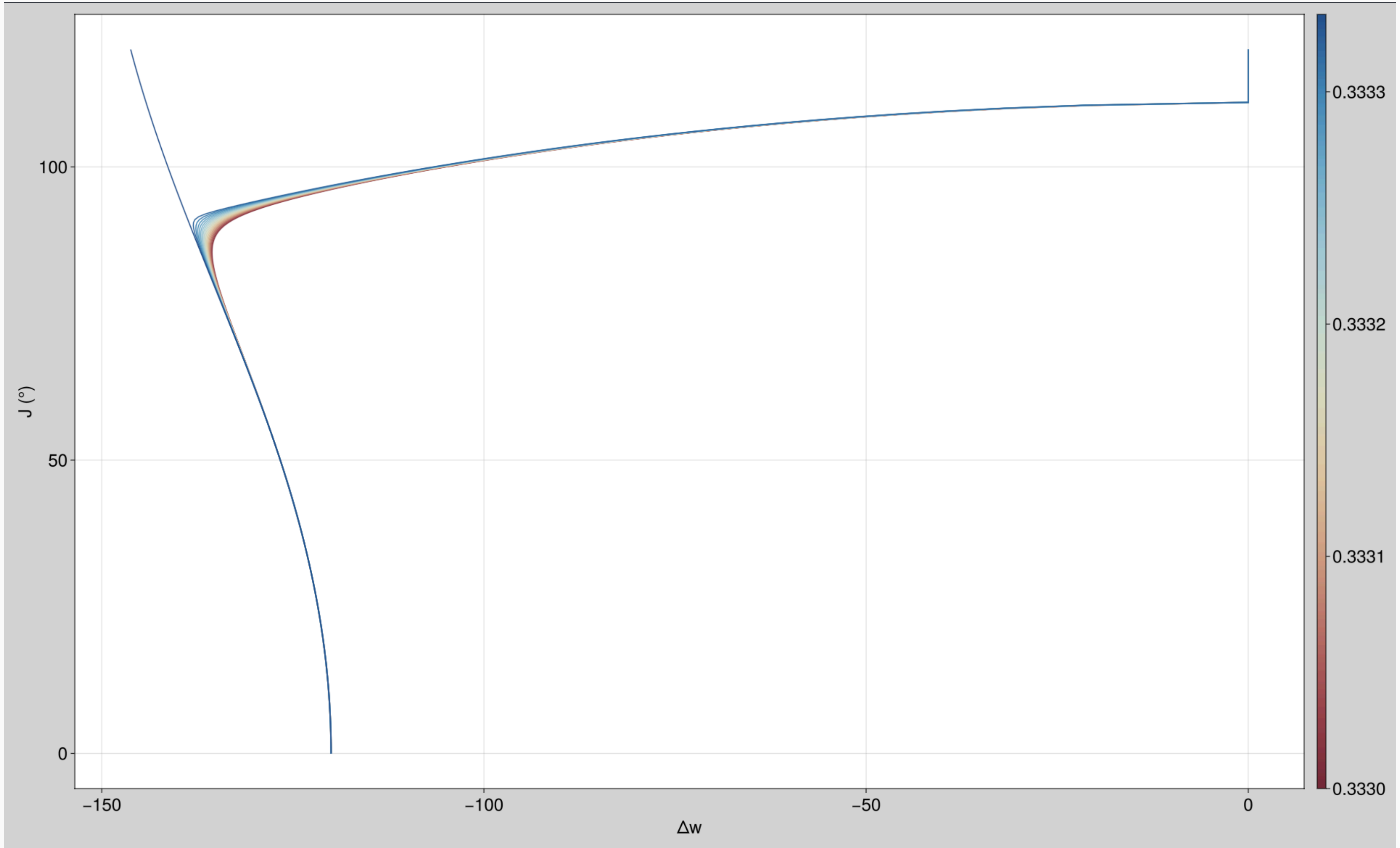
Similar to (Roberts, 2002)

Above Gascheau's value (long-term stability)



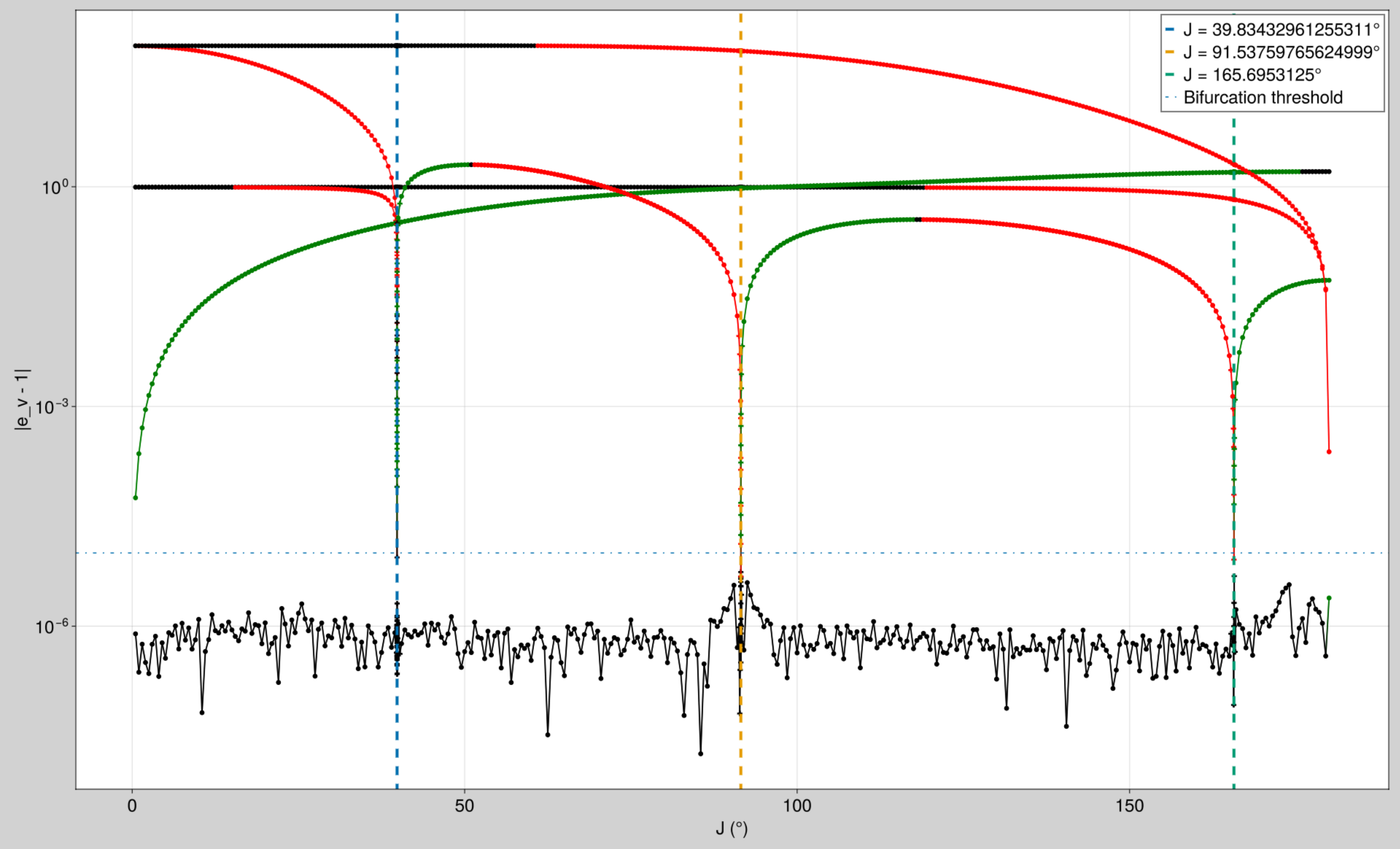


Approaching equal masses

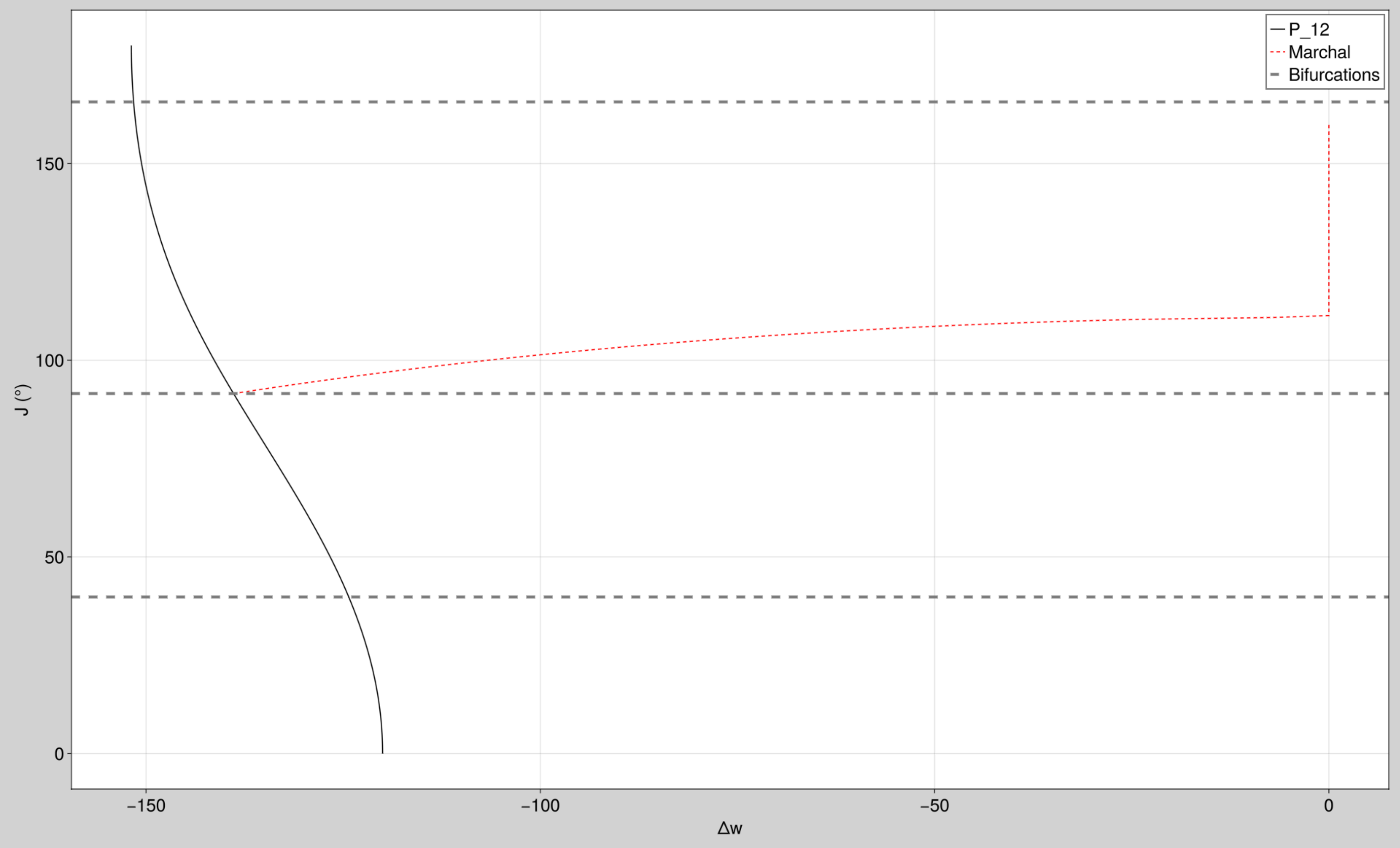


★

Bifurcations at equal masses

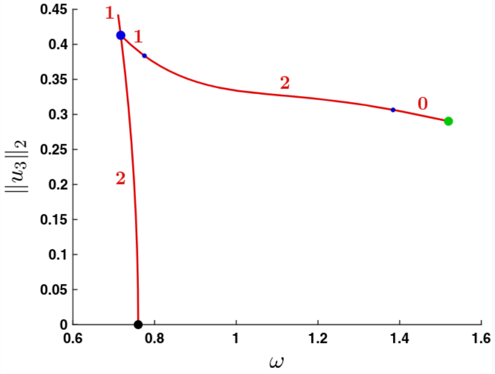


$\mathcal{VF}_L$ and P_{12}: Marchal's family



The other bifurcations

- Third bifurcation unexplored for now
- First bifurcation to a "lazy Eight"
  - Cf. (Calleja et al., 2021), and Hénot, Fejoz & Chenciner



Bifurcations at non-equal masses

These bifurcations also exist at low masses!

P_{12} confirmed at almost-equal masses ($\frac{1}{3} - \varepsilon \approx 10^{-5}$)

Bifurcations at non-equal masses

These bifurcations also exist at low masses!

P_{12} confirmed at almost-equal masses ($\frac{1}{3} - \varepsilon \approx 10^{-5}$)

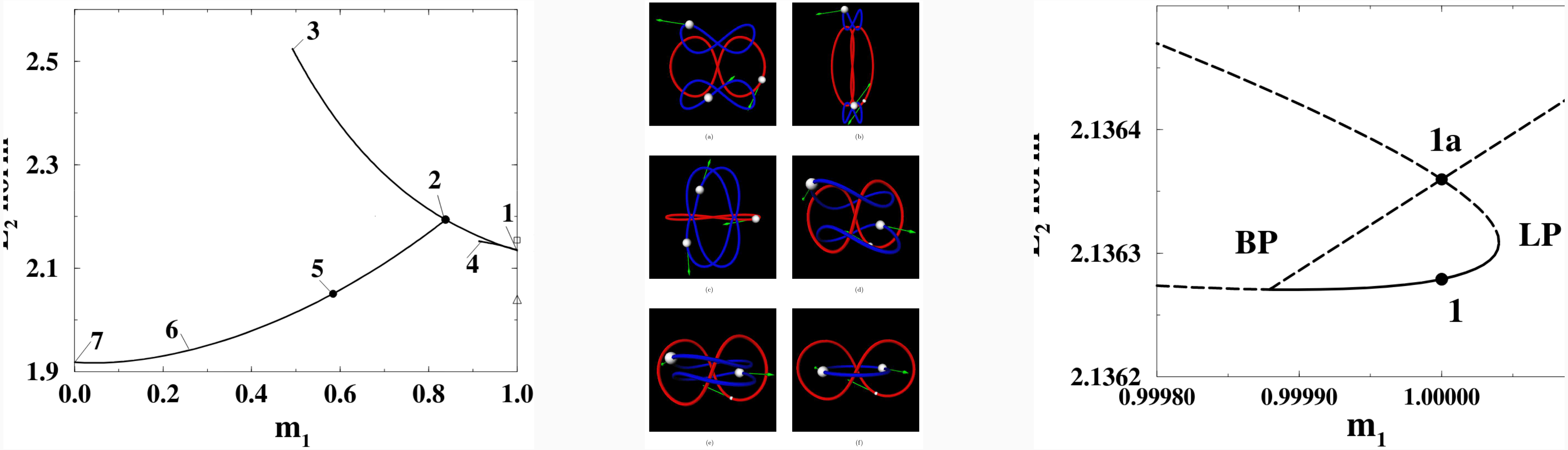


Figure 19: (Doedel et al., 2003)

Conclusion

Answering our initial questions

Q1: Does the family persist in the non-restricted/non-average problem?

- Yes

Q2:

Q3:

Q4:

Answering our initial questions

Q1: Does the family persist in the non-restricted/non-average problem?

- Yes

Q2: What is the stability of the orbits along this family?

- Stability changes, stability remnants beyond Gascheau

Q3:

Q4:

Answering our initial questions

Q1: Does the family persist in the non-restricted/non-average problem?

- Yes

Q2: What is the stability of the orbits along this family?

- Stability changes, stability remnants beyond Gascheau

Q3: Does it have an impact on the global stability of the co-orbital region?

- Yes

Q4:

Answering our initial questions

Q1: Does the family persist in the non-restricted/non-average problem?

- Yes

Q2: What is the stability of the orbits along this family?

- Stability changes, stability remnants beyond Gascheau

Q3: Does it have an impact on the global stability of the co-orbital region?

- Yes

Q4: How is it connected to P_{12}?

- Bifurcations

Answering our initial questions

Q1: Does the family persist in the non-restricted/non-average problem?

- Yes

Q2: What is the stability of the orbits along this family?

- Stability changes, stability remnants beyond Gascheau

Q3: Does it have an impact on the global stability of the co-orbital region?

- Yes

Q4: How is it connected to P_{12}?

- Bifurcations

Q1 to Q3: in an article, in review (Prieur and Robutel, 2026)

Future work

- Exploring bifurcations at equal masses
- Following P_{12} at non-equal masses
- Applying `TaylorInterface.jl` to other problems

Thank you for your attention!

Bibliography

Bezanson, J. et al. (2017) "Julia: A Fresh Approach to Numerical Computing," SIAM review, 59(1), pp. 65–98.

Calleja, R. et al. (2024) "From the Lagrange Triangle to the Figure Eight Choreography: Proof of Marchal's Conjecture." arXiv. Available at: https://doi.org/10.48550/arXiv.2406.17564.

Calleja, R. et al. (2021) "From the Lagrange Polygon to the Figure Eight I: Numerical Evidence Extending a Conjecture of Marchal," Celestial Mechanics and Dynamical Astronomy, 133(3). Available at: https://doi.org/10.1007/s10569-021-10009-9.

Chenciner, A. and Fejoz, J. (2008) "The Flow of the Equal-Mass Spatial 3-Body Problem in the Neighborhood of the Equilateral Relative Equilibrium," Discrete and Continuous Dynamical Systems - B, 10(2 & 3), pp. 421–438. Available at: https://doi.org/10.3934/dcdsb.2008.10.421.

Chenciner, A. and Montgomery, R. (2000) "A Remarkable Periodic Solution of the Three-Body Problem in the Case of Equal Masses," Annals of Mathematics, 152(3), pp. 881–901. Available at: https://doi.org/10.2307/2661357.

Doedel, E.J. et al. (2003) "Computation of Periodic Solutions of Conservative Systems with Application to the 3-Body Problem," International Journal of Bifurcation and Chaos, 13(6), pp. 1353–1381. Available at: https://doi.org/10.1142/S0218127403007291.

Jorba, À. and Zou, M. (2005) “A Software Package for the Numerical Integration of ODEs by Means of High-Order Taylor Methods,” Experimental Mathematics, 14(1), pp. 99–117. Available at: https://doi.org/10.1080/10586458.2005.10128904.

Leleu, A. (2016) Dynamique Des Planètes Coorbitales.

Marchal, C. (1990) The Three-Body Problem. Elsevier (Studies in Astronautics).

Marchal, C. (2009) “Long Term Evolution of Quasi-Circular Trojan Orbits,” Celestial Mechanics and Dynamical Astronomy, 104(1), pp. 53–67. Available at: https://doi.org/10.1007/s10569-009-9195-4.

Roberts, G.E. (2002) “Linear Stability of the Elliptic Lagrangian Triangle Solutions in the Three-Body Problem,” Journal of Differential Equations, 182(1), pp. 191–218. Available at: https://doi.org/10.1006/jdeq.2001.4089.